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RANDOM LÉVY MATRICES: II*

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We discuss the spectral density for standard and free random Lévy matrices in the large N limit. The eigenvalue spectrum is unbounded with power law tails in both cases.

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1. Introduction

The theory of random matrices belonging to the Gaussian universality class is well established. For example symmetric matrices whose independent elements are independent identically distributed (iid) random numbers with the normal distribution, or any random numbers with finite moments, belong to this class. In the large N limit the spectral properties of such matrices are identical to those of free Gaussian random matrices [1]. For symmetric

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matrices from the Lévy universality class the situation is different. Symmetric random matrices whose elements are iid Lévy numbers, have a different large N-limit than that of free random Lévy matrices [2]. We shall refer to the first class of matrices as standard random Lévy matrices to distinguish them from free Lévy matrices. The underlying measure in free random matrix theory is in general non-local, and only known explicitly for Cauchy and Lévy–Smirnov distributions. For a number of Lévy indices, the free eigenvalue distribution can be worked out explicitly exhibiting power law tails for large eigenvalues [2].

In a pioneering work on random matrix theory using iid Lévy generated entries of symmetric matrix, Cizeau and Bouchaud [3] have suggested that the distribution of eigenvalues of standard Lévy matrices follows a Lévy-like distribution whereby the range and the asymmetry of the distribution are determined self-consistently by pertinent integral equations at large N. We review and correct their arguments. In a separate section we will sketch the idea on how to derive the spectral density for free random Lévy matrices within the free random variable approach [2].

2. Random Lévy matrices

The absence of moments for Lévy distributions make the standard replica construction for matrices with random entries sampled from Lévy distributions mute. Instead, we follow Cizeau and Bouchaud [3] and define the $N \times N$ symmetric resolvent

$$G_{ij}^{N}(z) = (z - H)_{ij}^{-1} \,, \tag{1}$$

with entries $i, j = 1, ..., N$. Its $(N+1) \times (N+1)$ counterpart carry an extra row and column with entries $i, j = 0$. The extra diagonal entry obeys the recursive relation

$$S_0(z) = z - \frac{1}{G_{00}^{N+1}(z)} = H_{00} + \sum_{i,j}^{N} H_{0i} H_{0j} G_{ij}^{N}(z) \,. \tag{2}$$

Stability at large N demands that the matrix entries H_{ij}'s are $O(1/N^{1/\mu})$ and, therefore, H_{00} maybe omitted. When H_{ij} are sampled from an iid Lévy distribution,

$$P(H) \approx \frac{\mu \, R^{\mu}}{H^{1+\mu}} \tag{3}$$

the composition law for Lévy matrices implies that the diagonal contribution in (2) dominates over the off-diagonal one in large N. Thus, the self-energy satisfies

$$S_0(z) = z - \frac{1}{G_{00}^{N+1}(z)} \approx \sum_i^N H_{0i}^2 G_{ii}^N(z)\,. \tag{4}$$

which is exact in large N. Since the H_{0i} are sampled again from iid Lévy matrices, the sum of their squares in (4) is Lévy-like, *i.e.*

$$P_S(S) \propto L_{\mu/2}^{C(z),\beta(z)}(S)\,. \tag{5}$$

The parameters $C(z)$ and $\beta(z)$ are readily found from the composition rules for Lévy random variables[1]

$$\begin{aligned} C(z) = &\quad \frac{1}{N}\sum_i^N |G_{ii}|^{\mu/2}\,, \\ \beta(z) = &\quad \frac{\sum_i^N \mathrm{sgn}(G_{ii})|G_{ii}|^{\mu/2}}{\sum_i^N |G_{ii}|^{\mu/2}}\,. \end{aligned} \tag{6}$$

While each of the entries G_{00} and G_{ii} are different, in large N it is plausible that their distribution is Lévy-like. Since the self-energy is self-averaging we may use (5) and the measure

$$P_S(S)\,dS = P_S\left(z - \frac{1}{G_{00}}\right)\frac{dG_{00}}{G_{00}^2}\,. \tag{7}$$

to rewrite (6) in integral form

$$\begin{aligned} C(z) &= \int_{-\infty}^{\infty} dG|G|^{\mu/2-2} L_{\mu/2}^{C(z),\beta(z)}\left(z - \frac{1}{G}\right)\,, \\ \beta(z) &= \frac{\int_{-\infty}^{\infty} dG|G|^{\mu-2}\mathrm{sgn}(G) L_{\mu/2}^{C(z),\beta(z)}\left(z - \frac{1}{G}\right)}{\int_{-\infty}^{\infty} dG|G|^{\mu-2} L_{\mu/2}^{C(z),\beta(z)}\left(z - \frac{1}{G}\right)}\,. \end{aligned} \tag{8}$$

We may simplify (8) further by changing variables $1/G = C^{2/\mu}x'$ and $z = C^{2/\mu}z'$ and using the characteristic transform

$$L_{\mu/2}^{1,\beta}(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\exp\left(ikx - |k|^{\mu/2}(1 + i\beta\tau\mathrm{sgn}(k))\right)\,. \tag{9}$$

[1] The equation for the asymmetry $\beta(z)$ differs from the one quoted in [3].

The result is

$$C^2(z') \;=\; \frac{4}{\pi\mu}\Gamma\left(1-\frac{\mu}{2}\right)\sin\left(\frac{\pi\mu}{4}\right)\int\limits_0^\infty dp\cos\left(p^{2/\mu}z'-\beta(z')\;\tau\;p\right)e^{-p}\,,$$

$$\beta(z') \;=\; \tau^{-1}\frac{\int\limits_0^\infty dp\sin(p^{2/\mu}z'-\beta(z')\;\tau\;p)\,e^{-p}}{\int\limits_0^\infty dp\cos\left(p^{2/\mu}z'-\beta(z')\;\tau\;p\right)e^{-p}}\,. \tag{10}$$

with $\tau=\tan(\pi\mu/4)$. These equations can be solved self-consistently by first determining $\beta(z')$, then $C(z')$ and finally z from z' by the change of variable $z=C^{2/\mu}(z')z'$.

The distribution of entries G_{ii} yields the resolvent

$$G(z)=PV\int\limits_{-\infty}^{\infty}\frac{dx}{z-x}\,L^{C(z),\beta(z)}_{\mu/2}(x) \tag{11}$$

and, therefore, the eigenvalue distribution in integral form

$$\rho(\lambda)=\frac{1}{\pi^2}PV\int\limits_{-\infty}^{\infty}\frac{dz}{z-\lambda}\,G(z)\,. \tag{12}$$

The distribution of eigenvalues $\rho(\lambda)$ following from (12) is different from the one derived in [3]. The last equation can be combined with Eqs. (10) and (11) to numerically determine the eigenvalue distribution $\rho(\lambda)$. We found a perfect agreement of the distributions computed in this way with those obtained by numerical diagonalization of many randomly generated standard Lévy matrices. The comparison will be presented elsewhere.

3. Free random Lévy matrices

Following the original work by Voiculescu and coworkers [1], we have recently shown that the trace of the resolvent (1) for free random Lévy matrices obeys the transcendental equation

$$bG^{\mu}(z)-(z-a)G(z)+1=0\,, \tag{13}$$

with $b=C\,e^{i(\mu/2-1)(1+\beta)\pi}$ for $1<\mu<2$ and $b=C\,e^{i(\pi+(\mu/2)(1+\beta)\pi)}$ for $0<\mu<1$. The marginal case $\mu=1$ is given by a different transcendental equation

$$(z-a+i\gamma\,(1+\beta))\;G(z)+\frac{2\beta\gamma}{\pi}G(z)\ln\gamma\,G(z)-1=0\,. \tag{14}$$

The known solutions follow from $\mu = 1/2, 1, 2$. In particular, $\mu = 2$ yields the Gaussian resolvent

$$G_{\rm G}(z) = \frac{1}{2}\left(z - \sqrt{z^2-4}\right) , \tag{15}$$

while the marginal case $\mu = 1$ and $\beta = 0$ yields the Cauchy resolvent

$$G_{\rm C}(z) = \frac{1}{z-(a-i\gamma)} . \tag{16}$$

The discontinuity

$$\rho(\lambda) = -\frac{1}{\pi}{\rm Im}\, G(\lambda + i0) \tag{17}$$

is the standard density of states for free random Lévy matrices. Free randomness corresponds to coherent phase approximation (CPA) in large N which is a resummation of planar diagrams and a class of crossing graphs (non-planar).

4. Conclusion

We have reviewed and corrected the original arguments presented by Cizeau and Bouchaud for the density of states of symmetric and randomly generated Lévy matrix entries. A full numerical analysis of these results will be presented elsewhere. We have also reviewed the density of eigenvalues generated using free random variable calculus. Both ensuing spectra exhibit an unbounded support of eigenvalues in contrast to Gaussian random matrix theory. These results are of interest to a number of scale free systems ranging from networks to finances [4].

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